

Group actions and Helly's theorem

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Abstract

We describe a connection between the combinatorics of generators for certain groups and the combinatorics of Helly's 1913 theorem on convex sets. We use this connection to prove fixed point theorems for actions of these groups on nonpositively curved metric spaces. These results are encoded in a property that we introduce called "property FA_r ", which reduces to Serre's property FA when $r = 1$. The method applies to S -arithmetic groups in higher \mathbf{Q} -rank, to simplex reflection groups (including some nonarithmetic ones), and to higher rank Chevalley groups over polynomial and other rings (for example $SL_n(\mathbf{Z}[x_1, \dots, x_d])$, $n > 2$).

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1. Introduction

Serre proved [20,21] that $\Gamma = SL_3 \mathbf{Z}$ has *property FA*: any Γ -action on any simplicial tree T is *trivial*, i.e. it has a global fixed point. Serre's result was extended to arithmetic and S -arithmetic lattices in higher rank semisimple groups by Margulis [16], to groups with property T by Alperin and Watatani [2], and to many higher-rank Chevalley groups over certain commutative rings by Serre, Tits and Fukunaga [9].

These beautiful theorems unfortunately give only a rank one/higher rank dichotomy. While nontrivial actions of lattices in higher-rank simple Lie groups on trees do not exist at all, these groups have many nontrivial actions on higher-dimensional simplicial complexes of nonpositive

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curvature (in the CAT(0) sense). For example, the group $SL_n \mathbf{Z}[1/p]$ acts without a global fixed point on the affine building associated to $SL_n(\mathbf{Q}_p)$; this building is an $(n-1)$ -dimensional, nonpositively curved simplicial complex. Such actions were used by Quillen and Soulé (see [22]) to prove that such groups are pushouts of “ $(n-1)$ -simplices of groups”.

Since $SL_n \mathbf{Z}[1/p]$ acts (without a global fixed point) on a nonpositively curved simplicial complex of dimension $n-1$ but not one of dimension 1, it is natural to ask if it admits such an action in dimension m with $1 < m < n-1$. A related question: can $SL(n, \mathbf{Z}[1/p])$ be written as a (nice) pushout of a diagram of dimension $< n-1$? We prove below that the answer to these questions is “no”, but first it is useful to make the following:

Definition (*Property FA_n*). Let $n \geq 1$. A group Γ is said to have *property FA_n* if any isometric Γ -action on any n -dimensional, CAT(0) cell complex X has a global fixed point.

The property FA₁ corresponds with Serre’s property FA. If a group Γ has FA_n then it has FA_m for all $m < n$. In this paper, all complexes X are complete and have finitely many isometry types of cells, and all actions are by isometries. We emphasize that X need not be locally finite, and the putative action is allowed to have infinite stabilizers and to be nonfaithful. These possibilities are important for applications.

There is an even stronger notion which we shall find useful.

Definition (*Strong FA_n*). A group Γ is said to satisfy the *strong FA_n property* if any Γ -action on a complete CAT(0) space X satisfying the following properties has a global fixed point.

***n*-dimensionality:** The reduced homology group $\tilde{H}_n(Y) = 0$ for all open subsets $Y \subseteq X$.

Semisimplicity: The action of Γ on X is *semisimple*, i.e. the translation length (see below) of each $g \in \Gamma$ is realized by some $x \in X$.

Bridson proved (see [7]) that any isometric action on a cell complex as above must be semisimple, so that strong FA_n implies FA_n.

When Γ is an irreducible lattice in a semisimple Lie group, the problem of determining when a Γ -action on a singular CAT(0) space X must have a global fixed point has been studied for X a tree (Margulis [16]), X a Euclidean building (Margulis [17], Gromov and Schoen [11]), and X a CAT(−1) space (Burger and Mozes [5], Gao [10]). These papers have used either ergodic-theoretic or harmonic maps techniques. In this paper we introduce a new method, which reduces to Serre’s [20,21] in the case of trees. One advantage to our method is that it applies to many finitely-generated groups which are not lattices in any algebraic group.

1.1. Statement of results

Our first main result gives an extension of the Serre–Tits–Margulis theorems. For the definitions in the statement of the following theorem, see Subsection 4.1 below.

Theorem 1.1 (*Fixed point theorem for S-arithmetic groups*). Let k be an algebraic number field, and let G be an absolutely almost simple, simply-connected, connected, algebraic k -group. Suppose that $r = \text{rank}_k(G) \geq 2$. Let S be a finite set of valuations of k containing all the archimedean valuations. Let Γ be an S -arithmetic subgroup of G . Then Γ has the strong property FA_{r−1}.

As a special case we highlight the following.

Corollary 1.2. *Let $n \geq 3$. Then any finite index subgroup of $\mathrm{SL}_n \mathbf{Z}$ or of $\mathrm{SL}_n \mathbf{Z}[1/p]$ has the strong property FA_{n-2} .*

The dimension in Theorem 1.1 and Corollary 1.2 is sharp, as can be seen from the action of $\mathrm{SL}(n, \mathbf{Z}[1/p])$ on the affine building for $\mathrm{SL}(n, \mathbf{Q}_p)$. For $\mathrm{SL}(n, \mathbf{Z})$ the result is not sharp: an argument of Y. Shalom combined with some elementary arguments (see also [18]) gives that $\mathrm{SL}(n, \mathbf{Z}), n \geq 3$ has strong FA_n for all n . We also would like to emphasize that our method of proof uses unipotents in a strong way, so that (unlike many of the other methods) we can say nothing about the k -rank zero case.

Our next interest is in understanding actions of Chevalley groups over finitely generated commutative rings, for example $\mathrm{SL}(n, \mathbf{Z}[x_1, \dots, x_d])$. These groups are typically not (as far as we know) lattices in any algebraic k -group, and so the usual methods do not seem to apply.

For the basics of Chevalley groups, we refer the reader to [23,24]. Let Φ be a reduced, irreducible root system and let R be a commutative ring. Let $G(\Phi, R)$ be a Chevalley group of type Φ over R , and let $E(\Phi, R)$ denote its elementary subgroup. Our second main result extends Fukunaga's theorem [9] from FA_1 to FA_n .

Theorem 1.3 (Fixed point theorem for Chevalley groups). *Suppose Φ has rank $r \geq 2$. Then the group $E(\Phi, R)$ has the strong property FA_{r-1} .*

Since $\mathrm{SL}(n, \mathbf{Z}[x])$ surjects onto $\mathrm{SL}(n, \mathbf{Z}[1/p])$, the example above gives that the dimension in Theorem 1.3 is sharp.

It is a major problem in K -theory to determine when $G(R) = E(R)$. This is known for many rings, for example for polynomial or Laurent series rings over a field or over \mathbf{Z} (this is a famous theorem of Suslin). We thus obtain, for example, the following.

Corollary 1.4. *Let R be the ring of polynomials or Laurent series over \mathbf{Z} or over a field, and let $n \geq 3$. Then $\mathrm{SL}(n, R)$ has the strong property FA_{n-2} .*

Note that there is also overlap between the above theorems: for example $\mathrm{SL}_n(\mathbf{Z}[x])$ surjects onto $\mathrm{SL}_n(\mathbf{Z}[1/p])$, so the theorem for the former group implies it for the latter. However, Theorem 1.3 can be used to deduce the corresponding result for S -arithmetic lattices only in those G which are k -split.

After proving Theorem 1.3, we wondered if $G(\Phi, R)/E(\Phi, R)$ is always solvable when $\mathrm{rank}(\Phi) \geq 2$, and asked R. Hazrat and N. Vavilov, who then answered this question affirmatively in [15], as long as R has finite Bass–Serre dimension. Together with Theorem 1.3, this immediately gives the following.

Corollary 1.5. *Let $G(\Phi, R)$ be a Chevalley group of type Φ over a finitely generated commutative ring R with finite Bass–Serre dimension. Suppose Φ has rank $r \geq 2$. Then any action of $G(R)$ on an m -dimensional space satisfying the axioms of the strong property FA_m with $m < r$ factors through the action of the solvable group $G(R)/E(R)$.*

Our method also applies to certain reflection groups.

Theorem 1.6 (*Reflection groups*). *Let Γ_n be any discrete group generated by reflections in the sides of a finite volume n -simplex in \mathbf{R}^n or \mathbf{H}^n . Then Γ_n has the strong property FA_{n-1} .*

The dimension $n - 1$ in Theorem 1.6 is sharp. Also, the conclusion of this theorem is never true for polyhedra which are not simplices; groups generated by reflections in the faces of such polyhedra admit nontrivial actions on simplicial trees. A. Barnhill [4] has used the methods introduced in this paper to classify such actions, and to begin a classification for actions on higher-dimensional spaces. We will use Theorem 1.6 to prove that the Γ_n above, even nonarithmetic examples, have properties in common with arithmetic groups.

1.2. Integrality and finiteness

Representations of groups with property FA_n are quite constrained. Replacing “2” by “ n ” in 6.2.22 of [21], and noting that the affine building attached to GL_n of a field with discrete valuation is a nonpositively curved $(n - 1)$ -complex, immediately gives the following theorem.

Theorem 1.7 (*Integral eigenvalues*). *Suppose Γ has property FA_{n-1} . Let $\rho : \Gamma \rightarrow \text{GL}(n, k)$ be any representation of degree n over the field k . Then the eigenvalues of each of the matrices in $\rho(\Gamma)$ are integral. In particular they are algebraic integers if $\text{char}(k) = 0$ and are roots of unity if $\text{char}(k) > 0$.*

The general theory of groups with this property was introduced and studied by Bass [3], who called them groups of *integral n -representation type*. We thus have:

If Γ has FA_{n-1} then Γ is of integral n -representation type.

Bass proved a finiteness theorem (Proposition 5.3 of [3]) for groups of integral n -representation type, which immediately gives the following.

Theorem 1.8 (*Finiteness of the character variety*). *Suppose Γ has property FA_{n-1} . Then there are only finitely many conjugacy classes of irreducible representations of Γ into $\text{GL}_n(K)$ for any algebraically closed field K .*

By Theorem 1.6, groups of reflections in the faces of an $(n + 1)$ -simplex in \mathbf{H}^n satisfy the conclusions of Theorem 1.7 and Theorem 1.8. Note that there are nonarithmetic examples of such reflection groups acting on \mathbf{H}^n , $n = 3, 4, 5$ (see [25]). We also note that, for reflection groups, there are examples where the finite number in the conclusion of Theorem 1.8 can be made arbitrary large.

As a concrete example consider one of the reflection groups Γ_n ; it is a lattice in $\text{O}(n, 1)$ and has property FA_{n-1} . Hence there are only finitely many conjugacy classes of representation $\rho : \Gamma \rightarrow \text{O}(n, m)$ for $m < n$.

2. Group actions on CAT(0) spaces

2.1. Standing assumptions

Henceforth we will assume that X satisfies the properties as in the definition of strong property FA_n . So X is a metrically complete, n -dimensional, CAT(0) space, and all actions on X are isometric and semisimple.

2.2. Some CAT(0) preliminaries

We first recall some facts about CAT(0) spaces. A nice reference for background and definitions is [7].

For any $g \in \text{Isom}(X)$, let

$$\text{Minset}(g) = \{x \in X : d(x, gx) \leq d(y, gy) \forall y \in X\}$$

and let $\tau(g) = \inf_{x \in X} d(x, gx)$. The isometry g is called *semisimple* if the infimum in the definition of $\tau(g)$ is realized by some x . Under our standing assumptions, every $g \in \text{Isom}(X)$ is semisimple, and is one of two possible types: *elliptic*, which means that the fixed set $\text{Fix}(g) \neq \emptyset$; or *hyperbolic*, which means that $\text{Minset}(g) \neq \emptyset$ is the unique nonempty g -invariant set with every point $x \in \text{Minset}(g)$ translated a distance $\tau(g) > 0$.

For hyperbolic g , the set $\text{Minset}(g)$ may be realized as a union of parallel *axes* A_g , each of which is a g -invariant, bi-infinite geodesic line in X on which g acts by translation by $\tau(g)$. For elliptic or hyperbolic g , the set $\text{Minset}(g)$ is a closed, convex subset of X .

2.3. Finite group actions

One of the basic results on group actions on a complete CAT(0) space X is the following lemma.

Lemma 2.1 (*Bruhat–Tits fixed point theorem*). *Any finite group action on a CAT(0) space X has a global fixed point.*

The idea of the proof is that the orbit of any point under the group is a bounded set in X , hence has a unique center of mass which must be fixed by the group. For a detailed proof see, for example, [7].

2.4. General results on FA_n

We now record several elementary facts. Each holds with “ FA_n ” replaced by “strong FA_n ”.

1. If G has property FA_n then G has FA_m for all $m \leq n$.

Proof. Fixing a triangulation on each \mathbf{R}^n , one can easily convert an action on a CAT(0) m -complex X to an action on a CAT(0) n -complex $Y = X \times \mathbf{R}^{n-m}$. A global fixed point for the second action gives one for the first. \square

2. If G has FA_n then so does every quotient of G .

Proof. This is obvious. \square

3. Let H be a normal subgroup of G . If H and G/H have FA_n then so does G .

Proof. Suppose G acts on a $\text{CAT}(0)$ n -complex X . Then $\text{Fix}(H)$ is a nonempty convex subset of X , hence is a $\text{CAT}(0)$ m -complex for $m \leq n$. As H is normal, G acts on $\text{Fix}(H)$ and this action factors through G/H , which by hypothesis and (1) above has a fixed point $y \in \text{Fix}(H)$. Hence G fixes y . \square

4. If some finite index subgroup H of G has FA_n , then so does G .

Proof. Suppose G acts on a $\text{CAT}(0)$ n -complex X . Then $Hx = x$ for some $x \in X$. Let N be a finite index normal subgroup of G which is contained in H . So N fixes x and G/N acts on $\text{Fix}(N)$. As G/N is finite, we are done by Lemma 2.1. \square

Note that the converse to (4) is not true, even when $n = 1$; see, e.g. [21], 6.3.5.

2.5. Actions of nilpotent groups

In order to understand actions of more complicated groups on $\text{CAT}(0)$ spaces, we first need to understand actions of nilpotent groups. Actions of abelian groups by semisimple isometries on $\text{CAT}(0)$ spaces are completely understood. The key property of such group actions is that they have an invariant flat. Recall that a *flat* in X is an isometrically embedded copy of some Euclidean space \mathbf{R}^m , $m \geq 0$, in X .

The following proposition is well known; in this form it is (except for the last sentence, which we prove below) Theorem 7.20 of [7].

Proposition 2.2 (Abelian groups). *Let Γ be a finitely generated abelian group acting by semisimple isometries on a $\text{CAT}(0)$ space X . Then*

1. $\text{Minset}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Minset}(\gamma)$ is nonempty and splits as a product $Y \times \mathbf{R}^n$, $n \geq 0$ with Y nonempty and convex.
2. Γ leaves $\text{Minset}(\Gamma)$ invariant, preserves the product structure, acts trivially on the first factor, and acts cocompactly by translations on the \mathbf{R}^n factor. Further $n \leq \text{rank}_{\mathbf{Q}} \Gamma$.
3. Any $g \in \text{Isom}(X)$ which normalizes Γ leaves $\text{Minset}(\Gamma)$ invariant and preserves the isometric splitting $Y \times \mathbf{R}^n$. If g in fact centralizes Γ , then the induced action on the \mathbf{R}^n factor is by translations.

Proof. As mentioned above, we need only prove the last claim. By item (2), the action of g on the \mathbf{R}^n factor commutes with a cocompact group of translations. But any such isometry g of Euclidean space must itself be a translation: if A is the linear part of g then we would have $A(v + b) = Av + b$ for all $v \in \mathbf{R}^n$ and for every b in some basis, from which it follows that A is the identity. \square

Using Proposition 2.2, we can understand nilpotent groups acting on CAT(0) spaces. The following is likely well-known, and is a simple variation of the usual theorems on properly discontinuous actions; it was proven in the case of isometry groups of trees by Serre (see §6.5 of [21]). We include a proof here for completeness.

Proposition 2.3 (*Nilpotent groups*). *Let N be a finitely-generated, torsion-free nilpotent group acting by semisimple isometries on a CAT(0) space X . Then either*

1. *there is an N -invariant flat \mathbf{R}^n , $n > 0$ on which N acts by translations, hence factoring through an abelian group, or*
2. *$N \cdot x = x$ for some $x \in X$.*

Note that case (2) is simply case (1) with $n = 0$; we separate the cases in the statement of the proposition for pedagogical reasons.

Proof. We prove the proposition by induction on the length d of the lower central series for N . When $d = 1$ then N is abelian, and this is simply a restatement of Proposition 2.2, the case $n = 0$ giving a global fixed point.

Now consider an arbitrary nilpotent N with lower central series of length $d \geq 2$, and assume we know the proposition for nilpotent groups with lower central series of length less than d .

Apply Proposition 2.2 to the center $Z(N) \neq N$. So $\text{Fix}(Z(N)) = Y \times \mathbf{R}^n$, $n \geq 0$, where Y is a nonempty convex subset of X (hence is a complete CAT(0) space). As N centralizes $Z(N)$, it leaves $\text{Minset}(Z(N))$ invariant, preserves the product decomposition $Y \times \mathbf{R}^n$, and acts by translations on the \mathbf{R}^n factor. In other words, the N -action on $\text{Minset}(Z(N))$ is a product action $\rho = \rho_1 \times \rho_2$, where $\rho_1 : N \rightarrow \text{Isom}(Y)$ and $\rho_2 : N \rightarrow \text{Isom}(\mathbf{R}^n)$.

Since $\text{Fix}(Z(N)) = Y \times \mathbf{R}^n$, we have $\rho_1(Z(N)) = \text{Id}$, so ρ_1 factors through an $N/Z(N)$ action on the CAT(0) space Y . Note that $N/Z(N)$ is nilpotent of degree less than d , so we can apply the inductive hypothesis to this action. So $\rho_1(N/Z(N))$ leaves invariant some flat \mathbf{R}^m in Y on which it acts by translations (the case $m = 0$ being that $\rho_1(N/Z(N))$ has a global fixed point in Y). Hence $\rho(N) = \rho_1(N) \times \rho_2(N)$ leaves invariant the flat $\mathbf{R}^m \times \mathbf{R}^n$ in $\text{Minset}(Z(N))$, and acts on it via translations. \square

Proposition 2.3 gives the following simple criterion for an element in a nilpotent group N to have a fixed point, and for a nilpotent group N to have a global fixed point.

Corollary 2.4 (*Fixed points for nilpotent groups*). *Let N be a finitely-generated, torsion-free nilpotent group acting by semisimple isometries on a CAT(0) space X . Then*

1. *If $g^m \in [N, N]$ for some $m > 0$, then g has a fixed point.*
2. *If N is generated by elements each of which has a fixed point, then N has a global fixed point.*

Proof. Item (1) follows from item (1) of Proposition 2.3, since the translation subgroup of \mathbf{R}^n is abelian and so g must be in the kernel of the N -action on \mathbf{R}^n . For item (2), note that if (1) of Proposition 2.3 were to hold, then N would contain some generator of hyperbolic type, contradicting the given. Thus (2) of Proposition 2.3 holds, and we are done. \square

3. Helly's theorem

One of the most basic results of convexity theory is Helly's theorem.

Theorem 3.1 (Helly, 1913). *Let $\{X_i\}$ be any finite collection of nonempty, open convex sets in \mathbf{R}^n . If*

$$X_{i_1} \cap \cdots \cap X_{i_{n+1}} \neq \emptyset$$

for every $i_1 < \cdots < i_{n+1}$, then $\bigcap_i X_i \neq \emptyset$.

We will need a generalization of this theorem to convex subsets of CAT(0) n -complexes. The standard proof of Helly's theorem use *Radon's theorem*: any set of at least $n + 2$ points in \mathbf{R}^n can be divided into two nonempty subsets whose convex hulls intersect. However, Radon's theorem is *false* when \mathbf{R}^n is replaced by a CAT(0) n -complex X , as can be seen even when $n = 1$ and X is a tree.

However Helly's original proof, which seems to have been almost completely forgotten,² does generalize to this context. To state Helly's actual result in modern terminology, recall that a *homology cell* is a nonempty space X whose singular homology groups are isomorphic to those of a point, that is the reduced homology $\tilde{H}_q(X) = 0$ for all $q \geq 0$. Helly's original result and proof, restated in modern terminology by Debrunner (see Theorem 2 of [8]), is:

Theorem 3.2 (Topological Helly theorem). *Let X be a normal topological space with the property that every nonempty open subset $Y \subseteq X$ has $\tilde{H}_q(Y) = 0$ for all $q \geq n$. Let $\{X_i\}$ be any finite collection of nonempty closed homology cells in X . If the intersection of any r sets of this family is nonempty for $r \leq n + 1$, and is a homology cell for $r \leq n$, then $\bigcap_i X_i$ is a homology cell, in particular is nonempty.*

Note that nonempty convex subsets of \mathbf{R}^n are homology cells, as are intersections of convex sets. Hence Theorem 3.2 immediately implies Theorem 3.1.

Although Theorem 3.2 is stated in [8] only in the case when $X = \mathbf{R}^n$, the only property of \mathbf{R}^n used in his proof (see Lemma B_n of [8]) is that every nonempty open subset $Y \subseteq X$ has $\tilde{H}_q(Y) = 0$ for all $q \geq n$. Note that this property holds for all contractible, n -dimensional simplicial complexes, even ones that are not locally finite. The proof is exactly the same as the one given in [8] for \mathbf{R}^n .

Theorem 3.2 is stated in [8] for open sets X_i ; however, for a normal topological space, e.g. a metric space, the theorem implies the version for closed sets X_i . This is because, for a finite collection of closed sets in a normal topological space, there exist regular neighborhoods around each of the sets with the property that whenever two of the sets are disjoint so are their regular neighborhoods.

While Theorem 3.2 suffices for our applications, it is actually a very special case (when X is a CW complex) of the following old theorem of Leray (see, e.g. [6], Theorem VII.4.4).

² In fact a number of the “generalizations” of Helly's theorem in the literature follow immediately from Helly's original proof.

Theorem 3.3 (Leray). *Let $\{X_\alpha\}$ be a family of nonempty subcomplexes of a CW complex X . Suppose that every nonempty, finite intersection of the X_α is acyclic (i.e. has vanishing reduced homology groups). Then*

$$H_*\left(\bigcup_i X_i\right) = H_*(\mathcal{N}(\{X_i\}))$$

where $\mathcal{N}(\{X_\alpha\})$ is the nerve of the cover $\{X_\alpha\}$.

In particular, if $\{X_i\}$ is any collection of convex subsets of an n -dimensional metric space, and if we know the nerve of the covering has nonvanishing n -th homology, then further intersections are forced. Theorem 3.2 is the very special case when the nerve is the boundary of an n -simplex, in which case the full n -simplex must be part of the nerve, that is the sets have a common intersection. Thus Leray's theorem gives a vast generalization of Helly's original theorem.

It should be possible to apply the techniques of this paper to a much wider class of groups than we consider here, replacing our use of Helly's theorem by Leray's theorem.

4. Combinatorics of generating sets for S -arithmetic and Chevalley groups

4.1. Generating sets for S -arithmetic groups

We begin by recalling some definitions. An algebraic k -group G is *connected* if it is connected in the Zariski topology. Every algebraic k -group is an extension of a connected algebraic k -group by a finite group. G is called *absolutely almost simple* if any proper algebraic normal subgroup of G is trivial. A connected G is *simply connected* if every central isogeny $G' \rightarrow G$ from a connected algebraic k -group G' is an algebraic group isomorphism. The special linear group and the symplectic group are examples of absolutely almost simple, connected, simply-connected algebraic k -groups.

The goal of this subsection is to prove the following.

Theorem 4.1 (Combinatorics of generators). *Let k be an algebraic number field, and let G be an absolutely almost simple, simply-connected, connected, algebraic k -group. Suppose that $r = \text{rank}_k(G) \geq 2$. Let S be a finite set of valuations of k containing all the archimedean valuations. Let Γ be an S -arithmetic subgroup of G . Then there exists a set $C = \{\Gamma_1, \dots, \Gamma_{r+1}\}$ of finitely generated nilpotent subgroups of Γ which has the following properties:*

1. C generates a finite index subgroup of Γ .
2. Any proper subset of C generates a nilpotent subgroup of Γ .
3. There exists $m \in \mathbb{Z}^+$ so that for each $\Gamma_i \in C$ there is a nilpotent subgroup $N < \Gamma$ so that $r^m \in [N, N]$ for all $r \in \Gamma_i$.

Proof. For the basic facts about algebraic groups, k -split tori and root systems, we refer the reader to, e.g., Chapter I of [17] or Chapters IX and X of [13].

Fix a maximal k -split torus T in G , and let $X(T)$ denote the group of characters on T . Consider the adjoint action of T on the Lie algebra \mathfrak{g} of G . Let $\Phi = \Phi_k(T, G)$ be the system of k -roots with respect to T . For $\alpha \in X(T)$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}: \operatorname{Ad}(t)X = \alpha(t)X \ \forall t \in T\}.$$

As G is simple Φ is irreducible. Let $\Delta \subset \Phi$ be a system of simple roots and let Φ^+ (resp. Φ^-) be the corresponding system of positive (resp. negative) roots. For each $\alpha \in \Phi$, let U_α be the unique T -stable subgroup of G having Lie algebra the span of the root spaces $\{\mathfrak{g}_{r\alpha}: r \in \mathbb{Z}^+\}$. The group U_α is a unipotent k -subgroup of G , and is thus nilpotent.

Let \mathcal{O}_S be the ring of S -integers in k , and for any subgroup $H < G$ define $H(\mathcal{O}_S) := H \cap \operatorname{GL}(n, \mathcal{O}_S)$. For any ideal $\mathfrak{a} \neq 0$ in \mathcal{O}_S , let

$$H(\mathfrak{a}) = \{x \in H(\mathcal{O}_S): x \equiv \operatorname{Id} \pmod{\mathfrak{a}}\}. \quad \square$$

We will make essential use of the following theorem of Raghunathan, also proved by Margulis.

Theorem 4.2. (See [19], Theorem 1.2.) *With notation as above, let \mathfrak{a} be any nonzero ideal of \mathcal{O}_S . Then the group $E(\mathfrak{a})$ generated by $\{U_\alpha(\mathfrak{a}): \alpha \in \Phi\}$ has finite index in $G(\mathfrak{a})$.*

Now suppose we are given an S -arithmetic lattice Γ as in the hypothesis of the theorem. As Γ is S -arithmetic it contains $E(\mathfrak{a})$ for some \mathfrak{a} . By Theorem 4.2, this containment is finite index. By (4) on p. 1579, it suffices to prove the theorem for this finite index subgroup, which we will now denote by Γ .

Let $\Gamma_\alpha = U_\alpha(\mathfrak{a})$. Then Γ_α , being an S -arithmetic subgroup of the k -group U_α , is a finitely generated nilpotent group. Note that Γ_α is nilpotent since it is a subgroup of the nilpotent group U_α .

As Φ is irreducible, there is a unique longest root β' with respect to Δ . Let $\beta = -\beta'$. We now set

$$C = \{\Gamma_\alpha: \alpha \in \Delta\} \cup \{\Gamma_\beta\}.$$

Note that

$$|C| = |\Delta| + 1 = r + 1.$$

We claim that C has the required properties.

Proof of (1). We first show that C generates a finite index subgroup of $G(\mathfrak{a})$. For roots α, β , let

$$[\alpha, \beta] = \Phi \cap \{m\alpha + n\beta: m, n \in \mathbb{Z}^+\}.$$

We will need the following lemma, which is the lattice analog of a well-known fact about commutators of root groups in algebraic groups.

Lemma 4.3. (See [1], Proposition 7.2.4.) *For any $\alpha, \beta \in \Phi$, let N denote the group generated by Γ_α and Γ_β . Then the commutator subgroup $[N, N]$ contains a finite index subgroup of Γ_ϕ for each $\phi \in [\alpha, \beta]$.*

Let $\langle C \rangle$ denote the smallest subgroup of Γ containing each group in C . Since

$$\langle C \rangle \supset \Gamma_\Delta := \langle \{\Gamma_\alpha : \alpha \in \Delta\} \rangle$$

it follows from Lemma 4.3 and the fact that Δ generates Φ^+ under bracket that Γ_Δ contains a finite index subgroup Γ'_α of Γ_α for each $\alpha \in \Phi^+$.

We claim that $\langle C \rangle$ also contains a finite index subgroup Γ'_α of Γ_α for each $\alpha \in \Phi^-$. To prove this, first note that since $-\beta$ is the unique maximal root, we can write (see [12], 10.4 Lemma A)

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \quad \text{with each } k_\alpha < 0.$$

By definition, $\langle C \rangle$ contains (a finite index subgroup of) Γ_α for each $\alpha \in \Delta$, so by repeatedly applying Lemma 4.3 we see that $\langle C \rangle$ contains a finite index subgroup Γ'_α of Γ_α for each $\alpha \in [\beta, \Delta] \supseteq \{-\tau : \tau \in \Delta\}$, the final containment following from the fact that each $k_\alpha < 0$.

By the same argument as above which showed that the group generated by the $\Gamma'_\alpha, \alpha \in \Delta$ contains a finite index subgroup of each $\Gamma_\alpha, \alpha \in \Phi^+$, we deduce that $\langle C \rangle$ contains a finite index subgroup of each $\Gamma_\alpha, \alpha \in \Phi^-$, proving the claim.

Now for each $\alpha \in \Phi$, the finite index subgroup Γ'_α of the nilpotent group Γ_α determines an ideal \mathfrak{a}_α of the finitely-generated \mathbf{Z} -algebra of S -integers. Let \mathfrak{a}_Φ denote the intersection in the ring of S -integers of these finitely many ideals. Then the group generated by all of the $\Gamma'_\alpha, \alpha \in \Phi$ contains the group $\Lambda(\mathfrak{a}_\Phi)$ generated by all the subgroups of the U_α with entries in \mathfrak{a}_Φ . Hence $\langle C \rangle$ contains $\Lambda(\mathfrak{a}_\Phi)$. By Theorem 4.2, $\Lambda(\mathfrak{a}_\Phi)$ has finite index in Γ , and we have shown that C generates a finite index subgroup of Γ . \square

Proof of (2). To prove property (2), consider any proper subset Q of C . As $\langle Q \rangle$ is a subgroup of $\langle \{U_\alpha : \Gamma_\alpha \in Q\} \rangle$, it suffices to prove that this latter group is nilpotent. As the root groups corresponding to any collection of positive roots generate a unipotent subgroup of G (see, e.g., [17], p. 37), it suffices to prove the following. \square

Lemma 4.4. *Any subset of $|C| - 1$ elements of $\Delta \cup \{\beta\}$ is a subset of the positive roots with respect to some basis of the root space.*

Proof. By definition this holds for the subset Δ . So fix any $\alpha_i \in \Delta$ and consider the collection $B = (\Delta \setminus \alpha_i) \cup \beta$, which for convenience we will write as $\{v_1, \dots, v_r = \beta\}$.

Let $(,)$ denote the inner product on the root space. Note that B is a linearly independent set of vectors as $\{v_1, \dots, v_{r-1}\}$ is part of a basis and β has a nontrivial component in the direction of a vector (namely α_i) which completes that basis. Further we have that

$$(v_i, v_j) \leq 0 \quad \text{for all } i, j$$

since for $i \neq r$ we have $v_i \in \Delta$ and v_r is the negative of the maximal root and so has inner product ≤ 0 with each element of Δ . We now claim there is a regular vector γ such that $(\gamma, v_i) < 0$ for all i ; in other words all of the v_i 's lie *strictly* on the same side of the hyperplane γ^\perp orthogonal to γ .

Lemma 4.5. *Let $A = \{v_1, \dots, v_r\}$ be a set of linearly independent vectors in an r -dimensional inner product space $(V, (\cdot, \cdot))$. Suppose $v \in V$ is a nonzero vector with $(v, v_i) \leq 0$ for all i . Then there exists a vector γ with $(\gamma, v_i) < 0$ for each i . Further, γ can be chosen so that it does not lie in any subspace spanned by any proper subset of A .*

Proof. The lemma follows directly from the fact that for any basis $\{v, \dots, v_n\}$ of V , there is a linear functional f such that $f(v_i) > 0$; indeed, if $\{f_i\}$ is the dual basis to $\{v_i\}$, then we can take $f = \sum_{i=1}^n f_i$. Since the set of such f 's is clearly open, the first claim follows. The second claim is now clear. \square

Lemma 4.5 clearly implies the claim; the last claim of the lemma giving that γ is regular. Continuing with the proof of Lemma 4.4, for any regular vector γ let

$$\Phi^+(\gamma) = \{\alpha \in \Phi: (\alpha, \gamma) > 0\}.$$

Then (see, e.g. Theorem 10.1 of [12]) the set of all indecomposable roots in $\Phi^+(\gamma)$ is a root basis of Φ , so in particular every element of $\Phi^+(\gamma)$ is a positive root with respect to that basis. Choosing γ as above, we have shown that $B \subseteq \Phi^+(\gamma)$, and we are done. \square

Proof of (3). Let $\alpha \in \Delta \cup \{\beta\}$ be given. As Φ is irreducible and has rank at least 2, there exists $\sigma \in \Phi$ which is not proportional to α and so that $(\alpha, \sigma) > 0$. Then (see, e.g. [12], Lemma 9.4) $\alpha - \sigma \in \Phi$. Now the group N generated by Γ_σ and $\Gamma_{\alpha-\sigma}$ is nilpotent by the proof of (2), since any two nonproportional roots lie strictly on the same side of the hyperplane orthogonal to some regular vector; here again we are using that Φ is irreducible of rank at least 2. Further, by Lemma 4.3, the commutator subgroup $[N, N]$ contains a finite index subgroup of $\Gamma_{\sigma+\alpha-\sigma} = \Gamma_\alpha$. In particular some positive power of the generator $g_\alpha \in \Gamma_\alpha$ lies in $[N, N]$. \square

4.2. Chevalley groups over commutative rings

4.2.1. Basics of Chevalley groups

We now recall the definition and basic properties of Chevalley groups. The reader is referred to [14,23,24] for details.

Let R be any commutative ring (with unit) which is finitely-generated as a \mathbf{Z} -algebra. Let Φ be a reduced, irreducible root system, Δ a base for Φ , and Φ^+ the set of positive roots with respect to Δ . Each such Φ determines a unique Lie algebra $\mathfrak{g}(\Phi)$ over \mathbf{C} . Fixing a faithful, finite-dimensional, complex representation of $\mathfrak{g}(\Phi)$ determines a *Chevalley–Demazure group scheme*, or *Chevalley group*, $G(\Phi, R) = G_\rho(\Phi, R)$ over R . We always assume that $G(\Phi, \mathbf{C})$ is simply connected.

For each root $\alpha \in \Phi$ there is a group isomorphism

$$t \mapsto x_\alpha(t)$$

from R onto a subgroup X_α of $G(\Phi, R)$, called the *root subgroup* of $G(\Phi, R)$ corresponding to α . The group generated by $\{X_\alpha: \alpha \in \Phi\}$ is called the group of *elementary matrices*, and is denoted by $E(\Phi, R)$. We also let $U(\Phi, R)$ be the group generated by $\{X_\alpha: \alpha \in \Phi^+\}$. Then $U(\Phi, R)$ is nilpotent (see, e.g. [23,24]).

There are two fundamental formulas that hold in a Chevalley group. The first is elementary: for all $s, t \in R$ and $\alpha \in \Phi$ we have

$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t). \quad (4.1)$$

The second formula, which holds when Φ has rank at least 2, is the *Chevalley Commutator Formula*: for $s, t \in R$ and for linearly independent $\alpha, \beta \in \Phi$,

$$[x_\alpha(s), x_\beta(t)] = \prod_{i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j} s^i t^j) \quad (4.2)$$

where $i, j > 0$ and $N_{\alpha,\beta,i,j}$ are certain integers.

4.2.2. A nice generating set for $E(\Phi, R)$

We assume henceforth that Φ is reduced, irreducible, and has rank at least 2. Let $\{r_1, \dots, r_n\}$ be a generating set for R . As Φ is irreducible it has a unique maximal root β (see, e.g., Lemma 10.4.A of [12]). Let

$$C = \Delta \cup \{\beta\}.$$

Note that

$$|C| = \text{rank}(\Phi) + 1.$$

For each $\alpha \in \Delta \cup \{\beta\}$, let Γ_α denote the subgroup of $E(\Phi, R)$ generated by $\{x_\alpha(1), x_\alpha(r_1), \dots, x_\alpha(r_n)\}$.

The following proposition combines several results of Fukunaga [9].

Proposition 4.6 (*Combinatorics of generators for Chevalley groups*). *With notation as in the previous paragraph, the following holds.*

1. $E(\Phi, R)$ is generated by $\{\Gamma_\alpha: \alpha \in C\}$.
2. For each $\alpha \in C$ and each $r \in \Gamma_\alpha$, there exists $p = 1, 2, 3$ or 6 so that $r^p \in [U, U]$ for some nilpotent subgroup $U \subset E(\Phi, R)$.
3. For any proper subset $C' \subset C$, the group generated by $\{\Gamma_\alpha: \alpha \in C'\}$ is nilpotent.

Proof. Since Φ is irreducible and $\text{rank}(\Phi) \geq 2$, we may apply Proposition 2 of [9], which states that for each $t \in R$ and each $\alpha \in \Phi$, there exists $p = 1, 2, 3$ or 6 and some base Δ' so that $x_\alpha^p(t) \in [U, U]$, where $U = U(\Phi, \Delta')$, which is nilpotent. This immediately gives (2). We may also apply Proposition 4 of [9], which gives precisely (1). The proof of item (2) of Theorem 4.1 above, in particular Lemma 4.4, immediately gives (3). \square

5. Finishing the proofs of the main theorems

5.1. A general fixed point theorem

Our results on both S -arithmetic groups and Chevalley groups admit a common generalization.

Theorem 5.1. *Let Γ be a finitely generated group, and let $C = \{\Gamma_1, \dots, \Gamma_{r+1}\}$ be a collection of finitely generated nilpotent subgroups of Γ . Suppose that:*

1. *C generates a finite index subgroup of Γ .*
2. *Any proper subset of C generates a nilpotent group.*
3. *There exists $m > 0$ so that for any element r of any Γ_i there is a nilpotent subgroup $N < \Gamma$ with $r^m \in [N, N]$.*

Then Γ has the strong property FA_{r-1} .

Proof. Suppose that Γ acts on an $(r - 1)$ -dimensional CAT(0) complex X . By item (4) in Section 2.4, assumption (1) of the theorem gives that it suffices to prove that the subgroup generated by C , which we will by abuse call Γ , fixes some point of X .

By assumption (3) of the theorem, we may apply part (1) of Corollary 2.4 to give that $\text{Fix}(r) \neq \emptyset$ for each r in each Γ_i , and then we may apply part (2) of that corollary to give that $\text{Fix}(\Gamma_i) \neq \emptyset$ for each $1 \leq i \leq r + 1$. Hence we have a collection $\{\text{Fix}(\Gamma_i)\}$ of $r + 1$ nonempty, closed convex sets. As each $\text{Fix}(\Gamma_i)$ is convex it is contractible. Since C generates Γ it is enough to show that the intersection of these $r + 1$ sets is nonempty, thus giving a global fixed point for Γ .

Let U be any subgroup of Γ generated by any subset $Q \subset C$ of $|C| - 1 = r$ elements. By assumption (2), U is nilpotent. By the previous paragraph, U is generated by elements each of which has a fixed point. Corollary 2.4 now implies that $\text{Fix}(U)$ is a nonempty convex set.

We thus have $r + 1$ contractible sets $\{\text{Fix}(\Gamma_i)\}$ in the contractible space X , and the intersection of any r of these sets is nonempty. Applying the topological Helly theorem (Theorem 3.2) then gives $\bigcap_{i=1}^r \text{Fix}(\Gamma_i) \neq \emptyset$, and we are done. \square

5.2. Proof of Theorem 1.1 and Theorem 1.3

Theorem 1.1 follows immediately from Theorem 4.1 together with Theorem 5.1. Theorem 1.3 follows immediately from Proposition 4.6 together with Theorem 5.1.

5.3. Reflection groups

Let Γ_n be a discrete group generated by reflections in the faces of a compact n -simplex P in the space $Y = \mathbf{R}^n$ or $Y = \mathbf{H}^n$. As Γ_n is discrete, the subgroup H_v of Γ_n stabilizing any one of the $n + 1$ vertices v_i of P is a finite group. Note that the groups $\{H_v: v \in P\}$ generate all of Γ_n . Any proper subcollection of stabilizers of the v_i generates a group which stabilizes some face of P , and is therefore finite.

Now suppose that Γ_n acts on an m -dimensional space X with $m \leq n - 1$, satisfying the axioms in the definition of strong property FA_m . Then by the previous paragraph, and by the Bruhat–Tits fixed point theorem (Lemma 2.1 above), each stabilizer of a vertex v_i in P has nonempty fixed set, as does the intersection of the stabilizers of the vertices in any codimension one face of P . It follows that the nerve of the cover of fixed sets is an n -simplex. Since $\dim(X) = m < n$, it follows just as in the proof of Theorem 5.1 that the intersection of these fixed sets is nonempty, and we are done.

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